Fractals – Iterative Patterns on the Complex Plane

Another “operation” that can be performed with complex numbers is “iteration”. Iteration means the act of repeating a process usually with the aim of approaching a desired goal or target or result. Each repetition of the process is also called an “iteration,” and the results of one iteration are used as the starting point for the next iteration. Iteration in mathematics refers to the process of iterating a function (applying a function repeatedly) using the output from one iteration as the input to the next. (Wikipedia)

One famous iterative function is of the type $f(z) = z^2 + (0.2 + 0.2i)$. Following through the iterative process with any complex number $z$ will either result in the iterations spiraling down to a specific number (converge) or spiral up to infinity (diverge). Here’s some examples how it works:

### Converging Iteration:
Let $z = 1 + 1i$

1. $f(1+i) = (1+i)^2 + (0.2 + 0.2i) = 1 + 2i + 0.2 + 0.2i = 1.2 + 2.2i$
2. $f(1.2 + 2.2i) = (1.2 + 2.2i)^2 + (0.2 + 0.2i) = -4.6 + 1.08i$
3. $f(-4.6 + 1.08i) = (-4.6 + 1.08i)^2 + (0.2 + 0.2i) = 20.1936 - 9.736i$
4. $f(20.1936 - 9.736i) = (20.1936 - 9.736i)^2 + (0.2 + 0.2i) = 313.191785 - 393.0097792i$

...and after 40 more iterations it settles to $0.1273198322 + 0.3103616i$

### Diverging Iteration:
Let $z = 1 + 1i$

1. $f(1+i) = (1+i)^2 + (0.2 + 0.2i) = 1 + 2i + 0.2 + 0.2i = 1.2 + 2.2i$
2. $f(1.2 + 2.2i) = (1.2 + 2.2i)^2 + (0.2 + 0.2i) = -4.6 + 1.08i$
3. $f(-4.6 + 1.08i) = (-4.6 + 1.08i)^2 + (0.2 + 0.2i) = 20.1936 - 9.736i$
4. $f(20.1936 - 9.736i) = (20.1936 - 9.736i)^2 + (0.2 + 0.2i) = 313.191785 - 393.0097792i$

...and one can see that the number is getting larger and larger already....

Mathematicians were interested to see if there was any kind of pattern to the complex numbers that would converge to those that would diverge. They added a third dimension to the complex graph plane – color! If a complex number converged, its point on the complex plane was colored black. If a complex number diverged, it was given a specific color depending on how quickly it diverged. The resulting “fractal” images astounded mathematicians, and launched a whole new wave of mathematics, and even excites non-mathematicians. You might even see fractal landscapes in some video games! To speed up the tedious work of iterating hundreds of complex numbers, computers were programed to generate the graphs. Here are some examples of fractal designs:

For more on fractals, go to the World of Fractals page (http://www.angelfire.com/art2/fractals/index.html) or NOVA’s site (http://www.pbs.org/wgbh/nova/fractals/set.html). (Click-able links are on the class webpage under chapter 4.)

Try these on your calculator using the starting $z$ value given. Do they diverge or converge using $f(z) = z^2 + (0.2 + 0.2i)$? If they converge, what do they converge to?

1) $z = .3 + .3i$  
2) $z = .6 + .6i$  
3) $z = .8 + .8i$